CONCERNING CERTAIN TYPES OF CONTINUOUS CURVES[1]

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A point set M has property S[2] provided that for every positive number ϵ, M can be expressed as the sum of a finite number of connected point sets each of diameter less than ϵ. A point P of a continuum M is an end-point of M if and only if it is true that P is not a limit point of any connected subset of M minus any subcontinuum of M which contains P. If M is a continuous curve, the point P of M is an end-point provided no arc of M has P as an interior point. I have recently shown[3] that this latter definition for an end-point of a continuous curve is equivalent to the one given by Wilder.[4] The term acyclic continuous curve[5] will be used to designate a continuous curve which contains no simple closed curve.

THEOREM 1. *In order that a bounded continuum M should be a continuous curve every subcontinuum of which is a continuous curve it is necessary and sufficient that every connected open*[6] *subset of M should have property S.*

Proof. The condition is sufficient. Let M denote a bounded continuum every connected open subset of which has property S. Then since M itself is an open subset of M, M has property S, and by a theorem of Sierpinski's[7] it follows that M is a continuous curve. Suppose some subcontinuum N of M is not a continuous curve. Then by a theorem of R. L. Moore's,[8] there exist two concentric circles C_1 and C_2, $C_1 > C_2$, and a countable infinity of subcontinua of $N: N^*$, $N_1,N_2,N_3, \ldots\ldots\ldots$, such that (1) each of these continua has at least one point on each of the circles C_1 and C_2 and is a subset of the point set L consisting of the circles C_1 and C_2 together with all those points of the plane which lie between these circles, (2) each of these continua, save possibly N^*, is a maximal connected subset of the set of points common to N and L, and (3) N^* is the sequential limiting set of the sequence of continua $N_1,N_2,N_3,\ldots\ldots$ Let C_3 be a circle concentric with C_1 and C_2 and of radius $^1/_2$ the sum of the radii of C_1 and C_2. Then it is easily shown that there exists a countable infinity of continua T^*, $T_1,T_2,T_3,\ldots\ldots$, such that if L_0 denotes the set of points consisting of the circles C_1 and C_3 plus all those points of the plane which lie between these two circles, then (1) T^* is a subset of N^* and also of L_0, and for each positive integer n, T_n has at least one point on each of the circles C_1 and C_3 and is a maximal connected subset of the set of points common to N_n and L_0, and (2) T^* is the sequential limiting set of the sequence of continua $T_1,T_2,T_3,\ldots\ldots$ For each positive integer n, let X_n denote a point common to N_n and C_2. The set of points $X_1 + X_2 + X_3 + \ldots\ldots$ has a limit point A which belongs to both N^* and

C_2. Let ϵ denote the distance between C_2 and C_3. Since M is connected im kleinen, it is easily seen that the set of points $X_1 + X_2 + X_3 + \ldots\ldots$ contains a subsequence $X_{n_1}, X_{n_2}, X_{n_3}, \ldots\ldots$, such that (1) A is the sequential limit point of this sequence, and (2) for every positive integer i, X_{n_i} and A are the extremities of an arc t_i which is a subset of M and is of diameter less than $\epsilon/5i$. Let H denote the set of points $\sum_{i=1}^{\infty} t_i + \sum_{i=1}^{\infty} N_{n_i}$. Clearly H is connected.

Now let K denote the set of all points common to L_0 and N. For every positive integer i, T_{n_i} is a maximal connected subset of the closed point set K. Hence, by a theorem due to Zoretti,[9] there exists, for every i, a simple closed curve J_{n_i} which encloses T_{n_i} and contains no point of K and is such that every point of J_{n_i} is at a distance less than $\epsilon/2i$ from some point of T_{n_i}. Let K_M denote the set of points common to L_0 and M. For each integer $i > 0$, let I_i denote the set of points common to J_{n_i} and K_M. Let W denote the set of points $T^* + I_1 + I_2 + I_3 + \ldots\ldots$ Then W is a closed subset of M. Clearly W has no point in common with H. Let E denote the maximal connected subset of $M - W$ which contains H. Then clearly $M - E$ is closed. Hence E is a connected open subset of M, and by hypothesis must have property S. But since E contains infinitely many continua T_{n_i} all of diameter greater than half the distance from C_1 to C_3 and no two of which can be joined by a connected point set which is common to E and L_0, it easily follows by an argument very similar to that used by R. L. Moore to prove theorem 4 of his paper "Concerning Connectedness im kleinen and a Related Property"[10] that E does not have property S. Thus the supposition that M contains a subcontinuum which is not a continuous curve leads to a contradiction. Hence the condition is sufficient.

The condition is also necessary. Let N denote any definite connected open subset of a continuous curve every subcontinuum of which is a continuous curve, and let ϵ denote any definite positive number. Now unless N itself is of diameter $\leq \epsilon/4$, then N contains two points X_1 and Y_1 whose distance apart is $>\epsilon/4$. By a theorem of R. L. Moore's,[11] N contains an arc t_1 from X_1 to Y_1. Now H. M. Gehman has shown[12] that M cannot contain more than a finite number of mutually exclusive continua each of diameter greater than $\epsilon/4$. It readily follows that $N - t_1$ contains not more than a finite number of maximal connected subsets of diameter greater than $\epsilon/4$. Suppose it contains n such subsets. Let these be ordered $K_1, K_2, K_3, \ldots\ldots K_n$. Then by R. L. Moore's theorem quoted above, for each integer $i \leq n$, K_i contains an arc t_{i+1} of diameter $>\epsilon/4$. By Gehman's theorem it follows that $N - (t_1 + t_2 + \ldots + t_{n+1})$ contains at most a finite number of maximal connected subsets $H_1, H_2, H_3 \ldots\ldots H_m$ each of diameter $> \epsilon/4$. Again, for each integer $i \leq m$,

H_i contains an arc t_{n+1+i} of diameter $>\epsilon/4$. Let this process be continued. Since no two arcs t_i and t_j obtained in this way have a common point, and since each arc is of diameter greater than $\epsilon/4$, it follows by Gehman's theorem that this process must terminate after a finite number of steps. Thus we get a finite collection of arcs $t_1,t_2,t_3,\ldots.t_k$, such that every point of N is at a distance $\leq \epsilon/4$ from some one of these arcs. Let T denote the point set $t_1 + t_2 + t_3 + \ldots. + t_k$. Then since T has property S,[13] T is the sum of a finite number of connected point sets $k_1,k_2,k_3,\ldots.k_n$, each of diameter less than $\epsilon/4$. For each integer $i \leq n$, let l_i denote the set of all those points of N which can be joined to some point of k_i by a connected subset of N of diameter $\leq\epsilon/4$. Then for each integer $i \leq n$, l_i is connected and of diameter less than ϵ, and since every point of N is at a distance $\leq\epsilon/4$ from some set k_i, it follows that $N = l_1 + l_2 + l_3 + \ldots. + l_n$. Hence N has property S, and the theorem is proved.

THEOREM 2. *In order that a bounded continuum M should be an acyclic continuous curve it is necessary and sufficient that every connected subset of M should be uniformly connected im kleinen.*

Proof. The condition is sufficient. Let M denote any bounded continuum every connected subset of which is uniformly connected im kleinen. Evidently M must be a continuous curve. It must also be acyclic. For suppose it contains a simple closed curve J. Then if P denotes a point of J, the set $J - P$ is connected, but clearly it is not uniformly connected im kleinen, contrary to hypothesis. Hence M is an acyclic continuous curve.

The condition is also necessary. Let M denote any acyclic continuous curve, and let N be any connected subset of M. Suppose N is not uniformly connected im kleinen. Then there exists a positive number ϵ such that N contains two infinite sequences of points $X_1,X_2,X_3,\ldots.,$ and $Y_1,Y_2,Y_3,\ldots.,$ having the property that for each positive integer n, the distance from X_n to Y_n is less than $1/n$, but such that for no integer n can X_n be joined to Y_n by a connected subset of N of diameter less than ϵ. Now since M is uniformly connected im kleinen, there exists an integer k such that X_k and Y_k can be joined by an arc t of M of diameter less than $\epsilon/2$. And since, by a theorem of R. L. Wilder's,[14] every connected subset of an acyclic continuous curve is arcwise connected, N contains an arc t' from X_k to Y_k. But since t' must be of diameter greater than ϵ, t and t' cannot be identical. Hence their sum contains a simple closed curve. But this is contrary to our hypothesis that M is acyclic. Hence the condition of theorem 2 is necessary.

THEOREM 3. *In order that a bounded continuum M should be a continuous curve it is necessary and sufficient that the set of all the non-end-points of M should be uniformly connected im kleinen.*

Proof. The condition is sufficient. Let M denote a bounded continuum having the property that the set K of all the non-end-points of M is uniformly connected im kleinen. Now every point of $M - K$ is a limit point of K. Hence if K' denotes the set K plus all of its limit points, then $K' = M$. And by a theorem of R. L. Moore's,[15] since K is uniformly connected im kleinen, K' is connected im kleinen. Hence K', or M, is a continuous curve.

The condition is also necessary. Let K denote the set of all the non-end-points of a continuous curve. Since M is uniformly connected im kleinen, for every positive number ϵ there exists a positive number δ_ϵ such that every two points X_1 and X_2 of K whose distance apart is less than δ_ϵ are the extremities of an arc X_1X_2 of M of diameter less than ϵ. But since by definition, no end-point of M is interior to any arc of M, therefore every such arc X_1X_2 must be a subset of K. Hence K is uniformly connected im kleinen, and the theorem is proved.

THEOREM 4. *In order that a bounded continuum M should be a continuous curve it is necessary and sufficient that the set of all the non-end-points of M should have property S.*

Proof. The condition is sufficient. Let M be a bounded continuum having the property that the set K of all the non-end-points of M has property S. Let ϵ denote any definite positive number, and let K be expressed as the sum of a finite number of connected point sets $K_1, K_2, K_3 \ldots K_n$, each of diameter less than ϵ. Since every point of $M - K$ is a limit point of K, it follows that if K_i' denotes the set K_i plus all of its limit points, then $M = K_1' + K_2' + K_3' + \ldots + K_n'$. But for every integer $i \leq n$, K_i' is connected and of diameter less than ϵ. Hence M has property S, and by Sierpinski's theorem,[16] it is a continuous curve.

That the condition of theorem 4 is necessary follows from theorem 3 and from a theorem of R. L. Moore's[17] to the effect that every bounded point set which is uniformly connected im kleinen has property S.

THEOREM 5. *Every strongly*[18] *connected subset of a continuous curve every subcontinuum of which is a continuous curve is strongly connected im kleinen.*[18]

Proof. Let N denote any strongly connected subset of a continuous curve M every subcontinuum of which is a continuous curve. Suppose N is not strongly connected im kleinen at some one of its points P. Then there exists a circle C having P as center and an infinite sequence of points $X_1, X_2, X_3, \ldots$, all belonging to N and to the interior of C and such that P is the sequential limit point of this sequence but such that no one of these points can be joined to P by a subcontinuum of N which is contained in C plus its interior. Since N is strongly connected, X_1 lies together with P in a subcontinuum K_1 of N. And since every subcontinuum of M is a continuous curve, therefore X_1 and P can be joined by an arc t_1 which is a subset of N. Now t_1 must contain at least one point in the exterior of

C. In the order from X_1 to P on t_1, let Y_1 denote the first point belonging to C, and let a_1 denote the arc X_1Y_1 of t_1. Let X_{n_2} denote the first point of the sequence $X_1,X_2,X_3,\ldots.$ which does not belong to a_1. The set N contains an arc t_2 from X_{n_2} to P. In the order from X_{n_2} to P on t_2, let Y_2 denote the first point belonging to the closed set of points $C + a_1$, and let a_2 denote the arc $X_{n_2}\ Y_2$ of t_2. Let X_{n_3} be the first point after X_{n_2} of the sequence $X_1,X_2,X_3,\ldots.$ which does not belong to $a_1 + a_2$. Let a_3 be defined with respect to X_{n_3}, P, and the set $C + a_1 + a_2$ just as a_2 was defined with respect to X_{n_2}, P, and $C + a_1$. This process may be continued indefinitely; and thus we get a sequence of arcs $a_1,a_2,a_3,\ldots.$, such that (1) no two of them have an interior point of both in common, (2) each of them is a subset of N and of C plus its interior, (3) each of them contains a point of the sequence of points $X_1,X_2,X_3,\ldots.$, and (4) P belongs to their limiting set. Since every subcontinuum of M is a continuous curve, it follows from property (1) of this sequence and from a theorem of H. M. Gehman's[12] that for any $\epsilon > 0$, there are not more than a finite number of arcs of this sequence of diameter greater than ϵ. Hence no point other than P, which does not belong to one of the arcs of this sequence, can belong to their limiting set. Therefore, $P + a_1 + a_2 + a_3 + \ldots.$ is a closed point set. Not more than a finite number of arcs of the sequence $a_1,a_2,a_3, \ldots.$ can have points on C. Let $a_{k_1}, a_{k_2}, \ldots. a_{k_n}$ be the ones that do. For some integer $i \leq n$, a_{k_i} plus some infinite subcollection $a_{n_1},a_{n_2},a_{n_3}, \ldots.$ of the above sequence forms a connected point set. Let K denote this connected point set. Then since P is a limit point of the sequence of arcs $a_{k_i},a_{n_1},a_{n_2},a_{n_3},\ldots.$, therefore, P is a limit point of K, and the set of points $K + P$ is connected and closed. But $K + P$ is a subset of N and of C plus its interior, and it contains P and a point of the sequence of points $X_1,X_2,\ldots.$, contrary to our supposition. Thus the supposition that N is not strongly connected im kleinen leads to a contradiction.

THEOREM 6. *In order that the set of all the non-end-points of a continuous curve M should be a subset of the sum of the boundaries of the complementary domains of M it is sufficient that if A and B are any two points belonging to the same simple closed curve of M, then $M - (A + B)$ is not connected.*

Proof. Let E denote the set of all the non-end-points of a continuous curve M which satisfies the hypothesis of theorem 6. I have recently shown[3] that every point of E either is a cut point of M or else belongs to some simple closed curve of M. Let A denote any point of E. If A is a cut point of M, then by a theorem of R. L. Moore's,[19] A belongs to the boundary of some complementary domain of M. If A is not a cut point of M, then it belongs to some simple closed curve J of M. Now J contains at most a countable number of cut points of M.[20] Hence there exists, on J, a point B which is distinct from A and which is not a cut point of M. The set $M - B$ is connected. But by hypothesis $M - (A + B)$

is not connected. Hence A is a cut point of the set $M - B$. Let $(M - B) - A$ be expressed as the sum of two mutually exclusive sets S_1 and S_2 neither of which contains a limit point of the other. The sets $S_1 + A$ and $S_2 + A$ are connected. Now suppose, contrary to this theorem, that A does not belong to the boundary of any complementary domain of M. Then by a theorem of mine,[3] if X and Y denote points of S_1 and S_2, respectively, M contains a simple closed curve C which encloses A but which neither contains nor encloses any one of the points B, X and Y. Since $S_1 + A$ and $S_2 + A$ are connected sets, each containing a point within C and also a point without C, there must exist points P and Q, common to $S_1 + A$ and P and to $S_2 + A$ and C, respectively. Since C encloses A, P and Q must belong to S_1 and S_2, respectively. But C is a connected subset of $(M - B) - A$, and hence S_1 and S_2 are not mutually separated, contrary to hypothesis. Thus the supposition that A does not belong to the boundary of a complementary domain of M leads to a contradiction. Hence, in any case, A belongs to the boundary of some complementary domain of M; and since A is any point of E, then E must be a subset of the sum of the boundaries of the complementary domains of M.

Examples can easily be constructed to show that the condition of theorem 6 is not necessary.

[1] Presented to the American Mathematical Society, Sept. 9, 1926.

[2] Cf. Moore, R. L., "Concerning Connectedness im kleinen and a Related Property," *Fund. Math.*, **3**, 1921 (233–237).

[3] This theorem is in my paper, "Concerning Continua in the Plane," which has been submitted for publication in the *Trans. Amer. Math. Soc.*

[4] Wilder, R. L., "Concerning Continuous Curves," *Fund. Math.*, **7**, 1925 (358).

[5] This term has recently been introduced by Dr. H. M. Gehman.

[6] An open subset of a closed set M is any subset N of M such that $M - N$ is either vacuous or closed.

[7] "Sur une condition pour qu'un continu soit une courbe jordanienne," *Fund. Math.*, **1**, 1920 (44–66).

[8] "A Report on Continuous Curves from the Viewpoint of Analysis Situs," *Bull. Amer. Math. Society*, **29**, 1923 (296–297).

[9] Zoretti, L., "Sur les fonctions analytiques uniformes," *J. Math. pures appl.*, **1**, 1905 (9–11).

[10] Loc. cit.

[11] "Concerning Continuous Curves in the Plane," *Math. Zeit.*, **15**, 1922 (254–260).

[12] "Concerning the Subsets of a Plane Continuous Curve," *Annals of Math.*, **27**, 1925 (29–46).

[13] That T has property S is a direct consequence of Sierpinski's theorem mentioned in ref. 7.

[14] Loc. cit., p. 377, theorem 20.

[15] "Concerning Connectedness im kleinen and a Related Property," loc. cit., p. 233, theorem 1.

[16] Loc. cit.

[17] Moore, R. L., loc. cit., p. 234, theorem 3.

[18] A point set M is strongly connected provided that every two points of M lie to-

gether in a subcontinuum of M. A point set M is strongly connected im kleinen if for every point P of M and for every positive number ϵ there exists a positive number $\delta_{\epsilon p}$ such that every point X of M whose distance from P is less than $\delta_{\epsilon p}$ lies together with P in a *subcontinuum* of M of diameter less than ϵ. This definition for the term "strongly connected im kleinen" is identical with the one originally given by Hans Hahn [cf. *Weiner Akademie Sitzungsberichte*, **123**, Abt. IIa (2433–2489)], for the term "connected im kleinen." It has been customary, however, by R. L. Moore and others, to use the term "connected im kleinen" in the sense as given by the above definition with the words *connected subset* substituted for the word *subcontinuum*.

[19] "Concerning the Common Boundary of Two Domains," *Fund. Math.*, **6**, 1924 (211).

[20] Cf. Moore, R. L., "Concerning the Cut Points of Continuous Curves and of Other Closed and Connected Point Sets," these PROCEEDINGS, **9**, 1923 (101–106).

SUMMARY OF RESULTS AND PROOFS CONCERNING FERMAT'S LAST THEOREM (SECOND NOTE)

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As in the first note (these PROCEEDINGS, **12**, 1926, 106–9), we shall divide the discussion into two cases. If in the relation

$$x^p + y^p + z^p = 0, \tag{1}$$

with x, y, z, prime to each other, xyz is prime to the odd prime p, this will be referred to as case I, and if one of the integers x, y, z, is divisible by p, this will be called case II.

The theorems I–III, together with corollaries I and II to theorem II, of the first note (using the same notation), can be made much stronger by noting that we have from

$$\begin{aligned} &x + \beta^a y \equiv 0 \pmod{\mathfrak{p}}, \qquad x^p \equiv -\beta^{ap} y^p \pmod{\mathfrak{p}}, \\ &x^p = -y^p - z^p, \qquad y^p + z^p \equiv \beta^{ap} y^p \pmod{\mathfrak{p}}, \\ &y^p(\beta^{ap} - 1) \equiv z^p \pmod{\mathfrak{p}}. \end{aligned} \tag{2}$$

If r is a primitive root of n and we assume $n - 1 \not\equiv 0$ modulo p, the above congruence gives, observing that $\beta \equiv r \pmod{\mathfrak{p}}$,

$$y^p(r^{ap} - 1) \equiv z^p \pmod{\mathfrak{p}}.$$

Since $n - 1 \not\equiv 0$ modulo p, then there exists an unique integer $k < n - 1$ such that

$$r^{ap} - 1 \equiv r^k \pmod{n},$$